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CYCLE IN A SYSTEM CLOSE TO A RESONANCE SYSTEM†

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An autonomous system when the frequencies of the linear system satisfy a relation which is close, with a frequency detuning ε, to an exact two-frequency internal resonance is investigated. The fact that there is an isolated periodic solution in the general situation is established. The cycle occurs at a distance $O(\varepsilon)$ from zero in the case of third-order resonance and, at a distance of $O(\sqrt{\varepsilon})$ in the case of second- and fourth-order resonances. Systems of a general from as well as Lyapunov systems are considered. The problem of the stability of the cycles is studied. It is shown that, when small periodic perturbations of the order of μ act on the system being investigated, periodic motions exist in the μ^σ neighbourhood of zero for which $\varepsilon = O(\mu^k)$. In addition, $\sigma = 1/2$, $\mu \geqslant 1/2$ in the case of third-order resonance and $\sigma = 1/3$, $k \geqslant 2/3$ in the case of second- and fourth-order resonances.

1. PRELIMINARY REMARKS

The problem of the existence of local periodic motions in the neighbourhood of zero of an autonomous system has been fairly completely solved in the cases of Lyapunov system [1] and reversible systems [2, 3]. The existence of a pair of pure imaginary roots is common to both cases, and this leads to the existence of a one-parameter Lyapunov family which adjoins zero. The same family is found in reversible resonance systems [4], but as follows from the subsequent discussion, in the general situation, it does not occur in Lyapunov systems.

In systems of general form, the existence of local periodic motions cannot be deduced from the existence of a pair of pure imaginary roots; these motions only exist in strongly degenerate cases. This can be shown by analysing a second-order system.

In a system with two pairs of pure imaginary roots, a linear system allows of motion with two frequencies. For this reason, it is only in a situation close to a resonance case that the non-linear terms in the system lead to periodic oscillations in the neighbourhood of zero. Moreover, it can be expected that a cycle occurs.

Within the framework of the theory of the periodic motions of systems with a small parameter, the conclusions for Lyapunov systems [1] and reversible systems [2, 3] pertain to the rough cases [5] when the problem is solved in the zeroth approximation with respect to the small parameter and the perturbation belongs to the same class as the zeroth approximation. The problem of local periodic motions using a change of scale reduces to a problem containing a small parameter, which is successfully solved using the Lyapunov–Poincaré method. Hence, the existence of Lyapunov families in a Lyapunov system and reversible systems is established.

Systems of general form, which are close to resonance systems, belong to non-rough systems. These systems are studied by constructing [5] a new generating system which already contains a small parameter. The basic idea behind this approach is to pick out the most substantial perturbations from the point of view of the fact that periodic motions exist. The systematic development of the approach presented in [5] enables one to investigate systems including all resonance systems.

A general theorem [5, Theorem 5] has been established for non-rough systems of standard form. On carefully reading through the proof, it can be seen that the theorem yields the necessary and sufficient conditions for a periodic motion to exist if the case of multiple roots of the amplitude equation is excluded; only sufficient conditions are explicitly referred to in the formulation of the theorem. This fact is important when solving particular problems as it enables one to estimate the completeness of an investigation including that in this paper.

With the additional condition of existence of a first integral in the system being investigated, we obtain a problem on the local periodic motions of a Lyapunov system. Periodic motions in this system have

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been considered in the case of exact two-frequency resonance [6]. Here, only the case of a "fixed sign" integral was investigated if the resonance subsystem is taken into account at all. The case of a "non-fixed sign" integral is considered below. Using the integral, the order of the system is reduced by one and the resulting system depends on a parameter, the constant of the integral. In this case, a cycle exists on each energy level and forms a "family of cycle" in the initial system.

Note that Lyapunov systems have interesting applications in non-holonomic mechanics [7].

In a Hamiltonian system with two degrees of freedom (in the case of general position), the phase portraits (see [8, Chapter 8, Section 3.2]) give comprehensive information on the motion close to resonance. However, in the case of these mechanical systems, it is sometimes convenient to use equations in non-canonical variables (in quasi-coordinates, for example) and, in this sense, the results presented below are also useful.

Finally, we note that, when investigating a system which is close to a resonance system, it is convenient to use a normalizing transformation which is continuous with respect to a parameter ε (ε is a number which characterizes the detuning of the resonance) [9–11]. The basic idea behind such a transformation is clear: the non-resonance terms $C^{(j)}(\varepsilon)$ ($j = 2, 3, \ldots$), which become resonance terms when $\varepsilon = 0$, remain in the normal form of the system. On normalization, the required second-order terms of the initial system are retained in the quadratic terms and $C^{(2)}(\varepsilon) = C^{(2)}(0)$. If, however, 1:3 and 1:1 resonances are investigated, then the 1:2 resonance (with an accuracy of ε) is not present. Hence, there are no quadratic terms in the normal form and $C^{(3)}(\varepsilon) = C^{(3)}(0) + \varepsilon(\ldots)$. This means that only $C^{(3)}(0)$ can be taken into account in drawing conclusions on the basis of a study of the normal form of a system with an accuracy up to cubic terms inclusive.

2. CYCLE IN A SYSTEM CLOSE TO A RESONANCE SYSTEM

We will now consider a continuous system in the neighbourhood of the equilibrium and assume that the system of the first approximation has two pairs $\pm\lambda_s$ ($s = 1, 2$) of pure imaginary roots which satisfy (approximately) the resonance relation

$$\begin{aligned} &\lambda_1 + p\lambda_2 = i\kappa\varepsilon, \quad \kappa = \text{const}; \quad p = 1, 2, 3 \\ &\lambda_s = \lambda_s(0) + i\kappa_s\varepsilon, \quad \kappa_1 + p\kappa_2 = \kappa, \quad \kappa_{1,2} = \text{const} \end{aligned} \tag{2.1}$$

(ε is a number which characterizes the detuning of the resonance) and that the remaining roots are not multiples of the roots $\lambda_1(0)$, $\lambda_2(0)$. Simple roots are considered in the case of a 1:1 resonance.

Where $\varepsilon = 0$, the linear system admits of a family of periodic motions. It is also found that the whole resonance system allows of local periodic motions. However, this motion arises in the form of a cycle, and this cycle is located at a distance of $O(\varepsilon)$ or $O(\sqrt{\varepsilon})$ from the equilibrium position.

We introduce two pairs of complex-conjugate variables $z_s, \bar{z}_s$ ($s = 1, 2$) corresponding to the roots $\pm\lambda_s$ ($s = 1, 2$) and, on separating out the explicitly linear approximation, we obtain

$$\begin{aligned} &\dot{z}_s = \lambda_s z_s + Z_s(\mathbf{z}, \bar{\mathbf{z}}) + Z_s^*(\mathbf{z}, \bar{\mathbf{z}}, \mathbf{x}), \quad s = 1, 2 \\ &\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{X}(\mathbf{z}, \bar{\mathbf{z}}, \mathbf{x}), \quad \mathbf{x} \in \mathbf{R}^n \\ &Z_1^*(\mathbf{z}, \bar{\mathbf{z}}, \mathbf{0}) = Z_2^*(\mathbf{z}, \bar{\mathbf{z}}, \mathbf{0}) = \mathbf{0}, \quad \mathbf{A} = \mathbf{const} \end{aligned} \tag{2.2}$$

In the first group of equations (for $\mathbf{z}, \bar{\mathbf{z}}$), we put

$$\mathbf{x} = \mathbf{0}, \quad Z_s = Z_s^{(2)} + Z_s^{(3)} + Z_{s1}, \quad Z_{s1} = o(\|\mathbf{z}\|^3)$$

($Z_s^{(k)}$ are forms of order k) and we normalize the resulting system up to terms of the third order inclusive. We then apply this transformation to system (2.2) and, then, obtain a system containing the parameter ε

$$\begin{aligned} &\dot{\eta}_s = \lambda_s\eta_s + H_s^{(2)}(\boldsymbol{\eta}, \bar{\boldsymbol{\eta}}) + H_s^{(3)}(\boldsymbol{\eta}, \bar{\boldsymbol{\eta}}) + H_{s1}(\boldsymbol{\eta}, \bar{\boldsymbol{\eta}}) + \varepsilon H_{s2}(\varepsilon, \boldsymbol{\eta}, \bar{\boldsymbol{\eta}}) + H_s^*(\varepsilon, \boldsymbol{\eta}, \bar{\boldsymbol{\eta}}, \mathbf{y}), \\ &s = 1, 2 \\ &\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{Y}(\varepsilon, \boldsymbol{\eta}, \bar{\boldsymbol{\eta}}, \mathbf{y}), \quad \mathbf{y} \in \mathbf{R}^n \\ &H_{s2}(\varepsilon, \mathbf{0}, \mathbf{0}) \equiv \mathbf{0}, \quad H_s^*(\varepsilon, \boldsymbol{\eta}, \bar{\boldsymbol{\eta}}, \mathbf{0}) \equiv 0, \quad \mathbf{Y}(\varepsilon, \boldsymbol{\eta}, \bar{\boldsymbol{\eta}}, \mathbf{0}) = \mathbf{0}, \quad H_s^* = o(\|\boldsymbol{\eta}\|^3) \end{aligned} \tag{2.3}$$

(H_s are normal forms of order k).

We now change the scale in system (2.3): $(\mathbf{z}, \bar{\mathbf{z}}, \mathbf{x}) \to (\varepsilon_1\mathbf{z}, \varepsilon_1\bar{\mathbf{z}}, \varepsilon_1^\sigma\mathbf{x})$, $1 < \sigma < 2$, $\sigma = \text{const}$ and use the polar coordinates

$$\eta_s = \sqrt{r_s}\exp(i\theta_s), \quad \bar{\eta}_s = \sqrt{r_s}\exp(-i\theta_s), \quad s = 1, 2$$

As a result, we obtain the autonomous system

$$\begin{aligned}
\dot{r}_s &= \varepsilon_1 R_{s2}(\mathbf{r}, \theta) + \varepsilon_1^2 R_{s3}(\mathbf{r}, \theta) + \sqrt{r_s}(H_s^{**}e^{-i\theta_s} + \bar{H}_s^{**}e^{i\theta_s})\\
\dot{\theta}_s &= i\lambda_s + \frac{1}{2ir_s^{1/2}}[\varepsilon_1\Theta_{s2}(\mathbf{r}, \theta) + \varepsilon_1^2\Theta_{s3}(\mathbf{r}, \theta) + (H_s^{**}e^{-i\theta_s} - \bar{H}_s^{**}e^{i\theta_s})]; \quad s = 1, 2\\
\dot{\mathbf{y}} &= \mathbf{A}\mathbf{y} + \varepsilon_1^{-\sigma}\mathbf{Y}(\varepsilon, \varepsilon_1\sqrt{\mathbf{r}}\exp(i\theta), \varepsilon_1\sqrt{\mathbf{r}}\exp(-i\theta), \varepsilon_1^\sigma\mathbf{y})\\
H_s^{**} &= H_{s1} + \varepsilon H_{s2} + H_s^*\\
H_s^{**} &= H_s^{**}(\varepsilon, \varepsilon_1\sqrt{\mathbf{r}}\exp(i\theta), \varepsilon_1\sqrt{\mathbf{r}}\exp(-i\theta), \varepsilon_1^\sigma\mathbf{y}), \quad \theta = \theta_1 + p\theta_2
\end{aligned} \tag{2.4}$$

which is periodic with respect to the angles θ_1, θ_2 and contains the parameters ε and ε_1. The terms in ε_1, which are basic in the following account, are written out explicitly on the right-hand sides of system (2.4).

System (2.4) has the standard form for the application of the Lyapunov–Poincaré method in the theory of periodic motions [5]. We use the periodicity of system (2.4) with respect to the angle θ_2 and choose θ_2 as the new independent variable. Relation (2.1) enables as to introduce the resonance angle θ instead of the angle θ_1. As a result, we have an $(n + 3)$-order system which is periodic in θ_2 (and in θ), where the variables r_1, r_2 and θ play the role of amplitudes and the vector $\mathbf{y}$ plays the role of an "angle".

Theorem 1. A cycle

$$\begin{aligned}
z_1 &= \varepsilon_1\{(r_1^0 + O(\varepsilon_1))^{1/2}\exp[i(\theta^0 - p\theta_2 + O(\varepsilon_1))] + O(\varepsilon_1)\}\\
z_2 &= \varepsilon_1\{(r_2^0 + O(\varepsilon_1))^{1/2}\exp[i(\theta_2 + O(\varepsilon_1))] + O(\varepsilon_1)\}\\
\theta_2 &= [i\lambda_2 + O(\varepsilon_1)]t + \theta_2^0, \quad \theta_2^0 = \text{const}\\
\mathbf{x} &= \mathbf{O}(\varepsilon_1)
\end{aligned} \tag{2.5}$$

of system (2.2), where $\varepsilon_1 = \varepsilon$ when $p = 2$ and $\varepsilon_1^2 = \varepsilon$ when $p = 1$ or $p = 3$, corresponds to each simple root (r_1^0, r_2^0, θ^0) of the system of amplitude equations for the variables r_1, r_2, θ in the ε_1-neighbourhood of zero.

Proof. The existence of motions periodic in θ_2 of a system that is periodic in θ_2 and θ, which is obtained from the autonomous system (2.4), is guaranteed by the general theorem in [5]. These motions exist when $\varepsilon_1 < \varepsilon_0$, where ε_0 is a certain positive number and

$$r_s^0 = r_s^0 + O(\varepsilon_1), \quad \theta = \theta^0 + O(\varepsilon_1), \quad \mathbf{y} = \mathbf{O}(\varepsilon_1^\sigma)$$

Now, on taking account of the scaling and the autonomy of system (2.2), we obtain the cycle (2.5).

Remarks 1. Theorem 1 yields the conditions for the generation of an isolated cycle in the ε_1-neighbourhood of zero in a system of general form close to a resonant system.

2. It follows from Theorem 1 that, when analysing the problem, the investigation can be confined solely to a resonance system.

3. In the case of a system possessing the additional properties, such as a Hamiltonian system, a Lyapunov system or a reversible system, the amplitude equations have non-simple roots and Theorem 1 does not give the conditions for a cycle, that is, an isolated periodic motion. In a Hamiltonian system, there is a cycle in each system, constructed at a fixed energy level, and a "family of cycles" is formed in the initial system, that is, all of the periodic motions, strictly speaking, will be non-isolated. References to this can be found in [8]. A similar situation holds in systems with first integrals. Here, it is necessary to apply an assertion, similar to Theorem 1, to a system which has been reduced using first integrals. It is easy to derive the corresponding assertion, and it is used below in the treatment of a Lyapunov system. An assertion of the type of Theorem 1 also exists in the case of a reversible system.

3. THIRD-ORDER RESONANCE

Suppose we have $p = 2$ in relation (2.1). Then, in system (2.4), we obtain

$$\begin{aligned} \dot r_s &= 2\varepsilon R_s(\theta)\sqrt{r_1}r_2 + 2\varepsilon^2(A_{s1}r_1 + A_{s2}r_2)r_s + o(\varepsilon^2) \\ \dot\theta_s &= (3-2s)\omega_s + \varepsilon Q_s(\theta)r_1^{s-3/2}r_2^{2-s} + \varepsilon^2(B_{s1}r_1 + B_{s2}r_2) + o(\varepsilon^2) \\ \omega_s &= |\lambda_s|, \quad \theta = \theta_1 + 2\theta_2, \quad \omega_1 = 2\omega_2 + \kappa\varepsilon \\ R_s(\theta) &= a_s\cos\theta + b_s\sin\theta, \quad Q_s(\theta) = dR_s(\theta)/d\theta; \quad s = 1, 2 \end{aligned} \tag{3.1}$$

(a_s, b_s, A_{sj}, B_{sj} are real constants). We replace the two equations for θ_1 and θ_2 by a single equation for θ

$$\dot\theta = \varepsilon(\kappa + r_1^{-1/2}F) + \varepsilon^2(B_1r_2 + B_2r_2) + O(\varepsilon^2)$$

$$F = -(a_1r_2 + 2a_2r_1)\sin\theta + (b_1r_2 + 2b_2r_1)\cos\theta$$

$$B_1 = B_{11} + 2B_{21}, \quad B_2 = B_{12} + 2B_{22}$$

We then choose the angle θ_2 as the independent variable and write the system of amplitude equations in the first approximation in ε [5]

$$(a_s\cos\theta + b_s\sin\theta)\sqrt{r_1}r_2 = 0, \quad s = 1, 2, \quad F = 0 \tag{3.2}$$

Since $r_s \neq 0$ ($s = 1, 2$) for the motions being considered, we obtain that system (3.2) only has a solution in the case when

$$a_1/b_1 = a_2/b_2 = -\mathrm{tg}\theta^0 = -\xi \quad (\xi = \mathrm{const}) \tag{3.3}$$

Note that conditions (3.3) cannot be satisfied in an unstable resonance system and, when conditions (3.3) are satisfied, the system can be both unstable ($b_1b_2 > 0$) as well as stable in the second order ($b_1b_2 < 0$).

Suppose conditions (3.3) are satisfied. We then replace one of the two equations for r_s in system (3.1) by the equation

$$b_2\dot r_1 + b_1\dot r_2 = \varepsilon^2[b_2(A_{11}r_1 + A_{12}r_2)r_1 + b_1(A_{21}r_1 + A_{22}r_2)r_2] + o(\varepsilon^2)$$

and, consequently, the system of amplitude equations [5] takes the form

$$\begin{aligned} &\mathrm{tg}\theta = \xi \\ &\kappa + (b_1r_2 + 2b_2r_1)r_1^{-1/2}/\cos\theta = 0 \\ &A_{22}^*u^2 + A^*u + A_{11}^* = 0, \quad u = r_2/r_1 \\ &A^* = A_{12}^* + A_{21}^*, \quad A_{sj}^* = A_{sj}/b_s; \quad s, j = 1, 2 \end{aligned} \tag{3.4}$$

We will now analyse the compatibility of system (3.4). In degenerate cases

$$\begin{aligned} &\text{a) } A_{11}^*A_{22}^* < 0, \quad A^* = 0 \\ &\text{b) } A_{22}^* = 0, \quad A^*A_{11}^* < 0 \\ &\text{c) } A_{11}^* = 0, \quad A^*A_{22}^* < 0 \end{aligned} \tag{3.5}$$

the last equation of system (3.4) has a single positive root u^0. Then, $r_2^0 = u^0r_1^0$ and, from the second equation of system (3.4), we find

$$r_1^0 = -\kappa\cos\theta^0/(b_1 + 2b_2u^0)^2 \tag{3.6}$$

if $b_1 + 2b_2u^0 \neq 0$. The angle θ is determined from the first equation of (3.4) and, for a fixed value of the coefficient κ, the angle θ^0 has a unique value.

Consequently, in degenerate cases *a–c*, system (3.4) has a single simple root r_1^0, r_2^0, θ^0 and the root when $\kappa > 0$ necessarily corresponds to the root when $\kappa < 0$.

In the general cases

$$\begin{aligned} &\text{d) } D>0, \quad A^*A_{22}^*>0 \\ &\text{e) } D>0, \quad A^*A_{22}^*<0 \\ &D = A^{*2}-4A_{11}^*A_{22}^* \end{aligned} \tag{3.7}$$

system (3.4) has one or two simple roots respectively, and r_1^0 is found, as earlier, from the equality (3.6).

Theorem 2. In each of case *a–e*, which have been enumerated above, in a system close to a resonance system with the relation $\lambda_1 + 2\lambda_2 = \kappa\varepsilon$ (κ = const.), one cycle (cases *a–d*) or two cycles (case *e*) are generated in the ε-neighbourhood of zero, and a cycle necessarily exists both when $\kappa > 0$ as well as when $\kappa < 0$.

Remark. It follows from expression (3.6) that the cycle is generated in the neighbourhood of both a stable and unstable equilibrium point.

4. STABILITY OF CYCLES

We will now construct the equation in variations for the solution (r_1^0, r_2^0, θ^0) by putting

$$\Delta r_s = r_s - r_s^0, \quad s = 1, 2, \quad \Delta\theta = \theta - \theta^0$$

subject to the condition, of course, that one of conditions *a–e* in (3.5) and (3.7) is satisfied. We obtain

$$\begin{aligned} \Delta\dot{r}_1 &= -\varepsilon b_1\sqrt{r_1^0r_2^0}\chi\Delta\theta \\ \Delta\dot{r}_1/b_1 + \Delta\dot{r}_2/b_2 &= -2\varepsilon^2[(2A_1^*r_1^0 + A^*r_2^0)\Delta r_1^0 + (A^*r_1^0 + 2A_{22}^*r_2^0)\Delta r_2]/\omega_2 \\ \Delta\dot{\theta} &= \frac{-\varepsilon\chi}{2r_1^0\sqrt{r_1^0}}[(2b_2r_1^0 - b_1r_2^0)\Delta r_1 + 2b_1r_1^0\Delta r_2], \quad \chi = \frac{1}{\omega_2\cos\theta^0} \end{aligned} \tag{4.1}$$

The characteristic equation when $\varepsilon = 0$ only has zero roots $\rho = 0$. Hence, when $\varepsilon \neq 0$, we put $\rho = \varepsilon\rho_*/\omega_2$ and, in order to find ρ_*, we construct the equation

$$\det\begin{Vmatrix} -\rho_* & 0 & -b_1\sqrt{r_1^0r_2^0}\chi \\ -2\varepsilon(2A_{11}^*r_1^0 + A^*r_2^0) - \dfrac{\rho_*}{b_1} & -2\varepsilon(A^*r_1^0 + 2A_{22}^*r_2^0) - \dfrac{\rho_*}{b_2} & 0 \\ -\dfrac{\chi}{2r_1^0\sqrt{r_1^0}}(2b_2r_1^0 - b_1r_2^0) & -\dfrac{\chi b_1}{\sqrt{r_1^0}} & -\rho_* \end{Vmatrix} = 0$$

Expanding the determinant, we write this equation in the form

$$\begin{aligned} F &= [(1+\varepsilon a_1)\rho_*^2 + a_2]\rho_* + \varepsilon a_3 = 0 \\ a_2 &= (r_2^0b_1\chi)^2/(2r_1^0) \\ a_3 &= \chi^2(r_2^0/r_1^0)\left\{2b_1(A_{11}^* - A^*)r_1^{0^2} + [(2b_1 + b_2)A^* - 4b_1A_{22}^*]r_1^0r_2^0 + 2b_2A_{22}^*r_2^{0^2}\right\} \end{aligned} \tag{4.2}$$

(the explicit form of the coefficient a_1 is not subsequently required).

It can be seen from the graph of the function $F(\rho_*)$ (Fig. 1) that, when $a_3 = 0$ (the solid curve), Eq. (4.2) has a single zero root and a pair of pure imaginary roots. When $a_3 > 0$ ($a_3 < 0$), the zero root becomes a negative (positive) root (the dashed curves), and the imaginary pair of roots is split. We also note that the coefficient of ρ_*^2 is equal to zero and $a_2 > 0$. The application of the Routh–Hurwitz criterion reduces to just the single condition $a_3 > 0$.

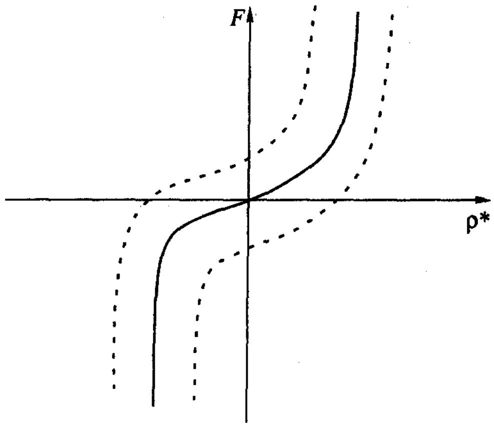

Fig. 1

Theorem 3. The cycle in the case of a third-order resonance ($\lambda_1 + 2\lambda_2 = \kappa\varepsilon$) is stable if

$$
\begin{aligned}
& A_{22}^* u^2 + A^* u^2 + A_{11}^* = 0, \quad u > 0 \\
& 2b_2 A_{22}^* u^2 + [(2b_1 + b_2)A^* - 4b_1 A_{22}^*]u + 2b_1(A_{11}^* - A^*) > 0
\end{aligned}
\tag{4.3}
$$

Remarks 1. It is obvious that conditions (4.3) are compatible both for a stable ($b_1 b_2 < 0$) as well as for an unstable equilibrium point.

2. Conditions (4.3) of Theorem 3 guarantee the asymptotic stability of the cycle of system (3.1) with respect to the variables r_1, r_2, $\theta = \theta_1 + 2\theta_2$. Since it is obvious that the cycle of system (3.1) is stable, it is sufficient to investigate it using just the parts of the variables r_1, r_2.

5. FOURTH-ORDER RESONANCE

We write out the normal form of the system when relation (2.1) with $p = 3$ holds

$$
\begin{aligned}
\dot{r}_s &= 2\varepsilon_1^2[(A_{s1}r_1 + A_{s2}r_2)r_s + R_s(\theta)r_1^{1/2}r_2^{3/2}] + o(\varepsilon_1^2) \\
\dot{\theta}_s &= (3 - 2s)\omega_s + \varepsilon_1^2[B_{s1}r_1 + B_{s2}r_2 + Q_s(\theta)r_1^{s-3/2}r_2^{5/2-s}] + o(\varepsilon_1^2) \\
\theta &= \theta_1 + 3\theta_2, \quad \omega_1 = 3\omega_2 + \kappa\varepsilon, \quad \varepsilon_1^2 = \varepsilon; \quad s = 1, 2
\end{aligned}
\tag{5.1}
$$

(A_{sj}, B_{sj}, a_s and b_s are real constants), and the functions $R_s(\theta)$ and $Q_s(\theta)$ are introduced by the last relations in (3.1). As in the case of a third-order resonance, we select the angle θ_2 as the new independent variable and, instead of the equation for θ_1, we use the equation for θ. The amplitude equations [5] for the resulting third-order system, which is periodic in θ_2, is then written in the form

$$
\begin{aligned}
& A_{s1}r_1 + A_{s2}r_2 + R_s(\theta)r_1^{s-3/2}r_2^{5/2-s} = 0, \quad s = 1, 2 \\
& \kappa + B_1 r_1 + B_2 r_2 + Q_1(\theta)r_1^{-1/2}r_2^{3/2} + 3Q_2(\theta)r_1^{1/2}r_2^{1/2} = 0 \\
& B_s = B_{1s} + 3B_{2s}, \quad s = 1, 2
\end{aligned}
\tag{5.2}
$$

We will now analyse the compatibility of system (5.2). The quadratic equation

$$
A_{22}u^2 + R_2(\theta)u + A_{21} = 0; \quad R_2(\theta) = a_2\cos\theta + b_2\sin\theta \tag{5.3}
$$

with the parameter θ with known constraints on the coefficients has one or two positive roots. In degenerate cases when one of the coefficients A_{11} or A_{22} is equal to zero, this constraint consists of the inequality $a_2^2 + b_2^2 < \max\{|A_{11}^2|, |A_{22}^2|\}$.

We now substitute the solution $u(\theta)$ of Eq. (5.3) into the first equation of system (5.2) and determine the root θ^0 from the equation

$$A_{11} + A_{12}u^2(\theta) + (a_1\cos\theta + b_1\sin\theta)u(\theta)\sqrt{u(\theta)} = 0$$

Finally, substituting $r_2 = u(\theta^0)r_1$ and $\theta = \theta^0$ into the last equation of system (5.2), we find r_1^0 and $r_2^0 = u(\theta^0)r_1^0$.

The algorithm which has been described proves the compatibility of system (5.2) and enables one to derive the conditions for a simple root to exist.

Theorem 4. A cycle corresponds to each simple root (r_1^0, r_2^0, θ^0) of a system of amplitude equations (5.2) in the $\sqrt{\varepsilon}$-neighbourhood of zero when the system is close to a resonance system ($\lambda_1 + 3\lambda_2 = \kappa\varepsilon$).

Remark. If follows from the third equation of system (5.2) that, in the case when $B_1 = B_2 = 0$, the existence of a cycle is possible both for $\kappa < 0$ and for $\kappa > 0$.

6. MULTIPLE ROOTS ($\lambda_1 + \lambda_2 = \kappa\varepsilon$)

We will now obtain the normal form of the system in the case of multiple roots

$$\begin{aligned}
\dot{r}_s &= 2\varepsilon_1^2\{(A_{s1}r_1 + A_{s2}r_2)r_s + [R_{s1}(\theta)r_1 + R_{s2}(\theta)r_2]\sqrt{r_1r_2} + \\
&+ (\alpha_s\cos2\theta + \beta_s\sin2\theta)r_1r_2\} + o(\varepsilon_1^2) \\
\dot{\theta}_s &= (3-2s)\omega_s + \varepsilon_1^2\{B_{s1}r_1 + B_{s2}r_2 + [Q_{s1}(\theta)r_1 + Q_{s2}(\theta)r_2]r_1^{s-3/2}r_2^{3/2-s} + \\
&+ (-\alpha_s\sin2\theta + \beta_s\cos2\theta)r_{3-s}\} + o(\varepsilon_1^2) \qquad (6.1)\\
R_{sj}(\theta) &= a_{sj}\cos\theta + b_{sj}\sin\theta, \quad Q_{sj} = dR_{sj}(\theta)/d\theta; \quad s, j = 1, 2 \\
\theta &= \theta_1 + \theta_2, \quad \omega_1 = \omega_2 + \kappa\varepsilon, \quad \varepsilon_1^2 = \varepsilon
\end{aligned}$$

(A_{sj}, B_{sj}, a_{sj}, b_{sj}, α_s, β_s are real constants). We write the amplitude equations for this system in the form

$$\begin{aligned}
&(A_{s1} + A_{s2}u^2)u^{s-1} + [(a_{s1} + a_{s2}u^2)\cos\theta + (b_{s1} + b_{s2}u^2)\sin\theta]u^{2-s} + \\
&+ (\alpha_s\cos2\theta + \beta_s\sin2\theta)u^{3-s} = 0, \quad s = 1, 2 \qquad (6.2)
\end{aligned}$$

$$\begin{aligned}
&a + r_1F(u, \theta) = 0 \qquad (6.3)\\
&F(u, \theta) = B_1 + B_2u^2 + \sum_{s=1}^{2}[(-(a_{s1} + a_{s2}u^2)\sin\theta + \\
&+ (b_{s1} + b_{s2}u^2)\cos\theta)u^{3-2s} + (-\alpha_s\sin2\theta + \beta_s\cos2\theta)u^{4-2s}] \\
&u = \sqrt{r_2/r_1}, \quad B_s = B_{1s} + B_{2s}, \quad s = 1, 2
\end{aligned}$$

It is obvious that the problem of the compatibility of system (6.2), (6.3) reduces to an analysis of system (6.2). A simple root (r_1^0, r_2^0, θ^0), $r_2^0, = u^0r_1$ of system (6.2), (6.3) corresponds to each simple root (u^0, θ^0), $u^0 > 0$ of system (6.2).

The following theorem therefore holds.

Theorem 5. A cycle, which is generated in the $\sqrt{\varepsilon}$-neighbourhood of zero, corresponds to each simple root (u^0, θ^0), $u^0 > 0$ of system (6.2).

Remark. The conditions obtained above in Theorems 1–5 essentially solve the problem of the existence of a cycle. A general theorem [5] is used here which yields the necessary and sufficient conditions, with apart from the absence of multiple roots of the system of amplitude equations. The results obtained are complete in this sense. However, use of the results requires an analysis of the roots of certain polynomials. It is found that, in special cases, the problem of a cycle is completely solved.

7. LYAPUNOV SYSTEM

We will investigate system (2.1), (2.2) with the additional condition of the existence of a first integral $V = h$ (h = const) in the neighbourhood of zero, where V is a smooth function. The existence of pure imaginary roots λ_1, λ_2 and the resonance condition (2.1) indicate that the quadratic part V_2 of the function V can be represented in the form

$$V_2 = \omega_1 r_1 - \omega_2 r_2 + W_2(\mathbf{x}), \quad \omega_s = |\lambda_s|, \quad s = 1, 2$$

(W_2 is a quadratic form of $\mathbf{x}$). We shall apply this case, which is the most interesting for applications, not only to equilibrium motions but also to steady motions. Note that the case of a "fixed sign" integral, when there is a "plus" sign in front of ω_2, has been considered earlier [6].

Suppose the remaining roots are not multiples of λ_2. Then, when $\varepsilon = 0$, the system has a Lyapunov family corresponding to the root λ_1 [1] and, when $\varepsilon \neq 0$, two families now exist corresponding to λ_1 and λ_2, that is, a second family arises. In the first family we have $r_1 = O(h)$, r_2, $||\mathbf{x}||^2 = o(h)$ and, in the second, $r_2 = O(h)$, $r_1 = O(h)$, $||\mathbf{x}||^2 = o(h)$. It is found that when $\varepsilon \neq 0$, a "family" of cycles arises in which $r_1 = O(r_2)$.

Below, we investigate a "family of cycles" in a system with a first integral. In this case, without loss in generality, we will consider the case when there is no additional system in the variable $\mathbf{x}$.

Taking account of scaling and relation (2.1), we represent the integral V in the form

$$(pr_1 - r_2)\omega_2 + \kappa\varepsilon r_1 + O(\varepsilon_1) = h^* \quad (h^* = \text{const}) \tag{7.1}$$

and consider motions for which $h^* = o(1)$. Then, the radius r_1 is determined from the integral (7.1) in terms of r_2 and, instead of the fourth-order system (2.4) ($y = 0$), we have a third-order system which depends on h^*. Next, we construct a second-order periodic system and apply the theorem from [5]. Then, on taking account of the fact that the integral (7.1) leads to definite relations between the coefficients of the normal form, we have formula (2.5) for the "family of cycles".

1. Third-order resonance. In accordance with the integral (7.1) in system (3.1), we have $2a_1 = a_2$, $2b_1 = b_2$. The system of periodic equations containing the cycle then has the form

$$\frac{dr_1}{d\theta_2} = \frac{1}{\Phi_2}[4\varepsilon(a_1\cos\theta + b_1\sin\theta)r_1^{3/2} + o(\varepsilon)] \tag{7.2}$$

$$\frac{d\theta}{d\theta_2} = \frac{1}{\Phi_2}\{\varepsilon[\kappa + 5r_1^{1/2}(-a_1\sin\theta + b_1\cos\theta)] + o(\varepsilon)\}$$

$$\Phi_2 = -\omega_2 + 2\varepsilon(-a_1\sin\theta + b_1\cos\theta)] + o(\varepsilon)$$

The system of amplitude equations always has the simple roots

$$5r_1^{0^{1/2}} = -\kappa b_1/(\chi_1\cos\theta^0), \quad \operatorname{tg}\theta^0 = -a_1/b_1, \quad \chi_1 = a_1^2 + b_1^2 \tag{7.3}$$

which proves the existence of a cycle that exists for any sign of κ. The characteristic equation for the cycle (7.3) acquires the form

$$\det\begin{Vmatrix} -\rho & -4\varepsilon\chi_1 r_1^{0^{3/2}}\cos\theta^0/(b_1\omega_2) \\ -5\varepsilon b_1/(2r_1^{1/2}\chi_1\omega_1\cos\theta^0) & -\rho \end{Vmatrix} = 0$$

and always has a pair of real roots of opposite sign. The cycle is always unstable.

Theorem 6. In a Lyapunov system which is close to a resonance system with the relations $\lambda_1 + 2\lambda_2 = \kappa\varepsilon$, a unique cycle always exists at each energy level $h = o(\varepsilon)$. The cycle is generated in the ε-neighbourhood of zero, it is unstable and is of a hyperbolic character.

Corollary. In a Hamiltonian system, which is close to a resonance system with the relation $\lambda_1 + 2\lambda_2 = \kappa\varepsilon$, a unique cycle

$$r_1 = r_1^0 + O(\varepsilon), \quad r_2 = 2r_1^0 + O(\varepsilon), \quad \theta_1 = -2\theta_2 + O(\varepsilon) + \theta_1^0$$

$$\theta_2 = -(\omega_2 + O(\varepsilon))t + \theta_2^0, \quad r_1^0 = (|\kappa b_1|/5\chi_1)^2, \quad \sin\theta^0 = 0, \quad \kappa\chi_1\cos\theta^0 < 0$$

$$\theta^0 = \theta_1^0 + 2\theta_2^0, \quad \omega_2 = |\lambda_2|, \quad \theta_2^0 = \text{const}$$

always exists at each energy level $h = o(\varepsilon)$, it is unstable and has a hyperbolic character.

Remark. In a Hamiltonian system, we have $a_1 = a_2 = 0$ [9].

2. *Fourth-order resonance*. In the case of a Lyapunov system, in relations (5.1) we have

$$A_{11} + 3A_{12} = A_{21} + 3A_{22}, \quad 3a_1 = a_2, \quad 3b_1 = b_2$$

and the periodic system of two equations takes the form

$$\frac{dr_1}{d\theta_2} = \frac{1}{\Phi_2}\{6\varepsilon[A_{11} + 3A_{12} + 3\sqrt{3}(a_1\cos\theta + b_1\sin\theta)]r_1^2 + o(\varepsilon)\}$$

$$\frac{d\theta}{d\theta_2} = \frac{1}{\Phi_2}\{\varepsilon[\kappa + B_1 + 3B_2 + 12\sqrt{3}(-a_1\sin\theta + b_1\cos\theta)r_1] + o(\varepsilon)\}$$

$$\Phi_2 = -\omega_2 + \varepsilon[B_{21} + 3B_{22} + 3\sqrt{3}(-a_1\sin\theta + b_1\cos\theta)]r_1 + o(\varepsilon)$$

It is clear that, when the condition

$$|A_{11} + 3A_{12}| \le 3\sqrt{3(a_1^2 + b_1^2)} \tag{7.4}$$

is satisfied, the first of the amplitude equations has the solution

$$\cos(\theta - \psi) = (A_{11} + 3A_{12})/\sqrt{27(a_1^2 + b_1^2)}, \quad \operatorname{tg}\psi = b_1/a_1 \tag{7.5}$$

Then, the value of r_1^0 is uniquely determined from the second equation

$$r_1^{0^2} = -\kappa/c > 0, \quad c = [B_1 + 3B_2 + 12\sqrt{3(a_1^2 + b_1^2)}\cos(\theta^0 + \psi)] \ne 0$$

if the denominator c is non-zero. The existence of a cycle is thereby proved.

We will now construct the characteristic equation of the system in variations in the neighbourhood of the cycle

$$\det\begin{Vmatrix} -\rho & -18\varepsilon r_1^{0^2}\sqrt{3(a_1^2 + b_1^2)}\cos(\theta^0 + \psi)/\omega_2 \\ -\varepsilon\kappa/(\omega_2 r_1^0) & -\rho \end{Vmatrix} = 0$$

from which it follows that the stability depends on the sign of the expression

$$k = -c\cos(\theta^0 + \psi) \tag{7.6}$$

When $k > 0$, the cycle has a hyperbolic character and, when $k < 0$, the cycle is stable.

Theorem 7. When (7.4) is satisfied, a cycle in a Lyapunov system in a situation close to resonance ($\lambda_1 + 3\lambda_2 = \varepsilon\kappa$) is generated in the $\sqrt{\varepsilon}$-neighbourhood of zero. The cycle is defined by formulae (7.5) and (7.6), it has a hyperbolic character when $k > 0$ and is linearly stable when $k < 0$.

Remark. In a Hamiltonian system, we have $A_{11} = A_{12} = a_1 = 0$ [9].

3. *Multiple roots.* We will now consider the situation when $\lambda_1 + \lambda_2 = \kappa\varepsilon$ in a Lyapunov system. Then, in accordance with the integral (7.1) in system (6.1), we obtain

$$A_{1j} = A_{2j},\quad a_{1j} = a_{2j},\quad b_{1j} = b_{2j},\quad \alpha_1 = \alpha_2,\quad \beta_1 = \beta_2,\quad (j = 1, 2)$$

For these conditions, we write the periodic system

$$\begin{aligned} \frac{dr_1}{d\theta_2} &= \frac{2\varepsilon}{\Phi_2}[(A + a_*\cos\theta + b_*\sin\theta + \alpha_1\cos 2\theta + \beta_1\sin 2\theta)r_1^2 + o(\varepsilon)] \\ \frac{d\theta}{d\theta_2} &= \frac{\varepsilon}{\Phi_2}\{\kappa + [B + 2(-a_*\sin\theta + b_*\cos\theta - \alpha_1\sin 2\theta + \beta_1\cos 2\theta)r_1] + o(\varepsilon)\} \end{aligned} \tag{7.7}$$

$$\Phi_2 = -\omega_2 + \varepsilon[B_2 - a_2\sin\theta + b_2\cos\theta - \alpha_2\sin 2\theta + \beta_2\cos 2\theta]r_1 + o(\varepsilon)$$

$$A = A_{11} + A_{12},\quad a_* = a_{11} + a_{12},\quad b_* = b_{11} + b_{12},\quad B = \sum_{s, j = 1}^{2} B_{sj}$$

$$B_2 = B_{21} + B_{22},\quad a_2 = a_{21} + a_{22},\quad b_2 = b_{21} + b_{22}$$

The system of amplitude equations

$$R(\theta)r_1^2 = 0,\quad \kappa + F(\theta)r_1 = 0$$

$$R = A + \mathrm{Re}(c_* e^{-i\theta}) + \mathrm{Re}(\gamma e^{-2i\theta})$$

$$F = B + 2[\mathrm{Im}(c_* e^{-i\theta}) + \mathrm{Im}(\gamma e^{-2i\theta})]$$

$$c_* = a_* + ib_*,\quad \gamma = \alpha_1 + i\beta_1$$

has the solution (r_1^0, θ^0), $r_1^0 > 0$, if $R(\theta^0) = 0$, $F(\theta^0) \neq 0$. This means that a cycle exists if $R(\theta) = 0$ for certain θ.

Note that the problem of finding the roots of the trigonometric expression for specified coefficients $A, a_*, b_*, \alpha_1, \beta_1$ does not present any difficulty.

We will now consider the problem of the stability of the cycle which has been constructed and construct equations in variations for the solution (r_1^0, θ^0). We then write the characteristic equation and calculate the roots

$$\rho^2 = -\varepsilon^2 cR'(\theta^0)/(r_1^0\omega_2^2)$$

$$c = B + R'(\theta^0),\quad R'(\theta^0) = -a_*\sin\theta^0 + b_*\cos\theta^0 - 2\alpha_1\sin 2\theta^0 + 2\beta_1\cos 2\theta^0$$

The stability when $cR'(\theta^0) > 0$ and instability when $cR'(\theta^0) < 0$ follows from this.

Theorem 8. In a Lyapunov system which is close to a resonance system with the relation $\lambda_1 + \lambda_2 = \varepsilon\kappa$, a cycle is generate in the $\sqrt{\varepsilon}$-neighbourhood of zero, if the equation $R(\theta) = 0$ has a root θ^0 for which $F(\theta^0) \neq 0$. The cycle is linearly stable when $cR'(\theta) > 0$ and is of a hyperbolic character when $cR'(\theta^0) < 0$.

Remark. In a Hamiltonian system, we have $A = a_* = \alpha_1 = 0$ [9].

8. THE ACTION OF PERIODIC PERTURBATIONS

We will assume that small periodic perturbations of the order of μ act on an autonomous system in the neighbourhood of zero. Forced oscillations of amplitude $O(\mu^\sigma)$, $\sigma > 0$ would then be expected in the general case. We will now investigate how non-linearities affect this amplitude. This problem, together with the problem of the existence of oscillations, is solved by applying the general theorem from [5]. In a periodic system, a resonance situation arises [1] when the system has even just one frequency close to the frequency of the perturbation. In order to elucidate the essence of the matter here, it is possible to confine ourselves to analysing just the second-order system

$$\begin{aligned} \dot{x} &= -\omega y + X(x, y) + \mu X_1(\mu, x, y, t) \\ \dot{y} &= \omega x + Y(x, y) + \mu Y_1(\mu, x, y, t), \quad \omega = p + \kappa\varepsilon, \quad p \in \mathbf{N} \end{aligned} \tag{8.1}$$

(X and Y are non-linear terms, X_1 and Y_1 are 2π-periodic functions of t and κ = const.)
We carry out the scaling: $(x, y) \to (\varepsilon_1 x, \varepsilon_1 y)$. Then, in the general case, we have

$$X(\varepsilon_1 x, \varepsilon_1 y) = \varepsilon_1^3 X_0(\varepsilon_1, x, y), \quad Y(\varepsilon_1 x, \varepsilon_1 y) = \varepsilon_1^3 Y_0(\varepsilon_1, x, y), \quad \varepsilon = \varepsilon_1^k (k \geq 2)$$

We now put $\varepsilon_1 = \mu^{1/3}$ and write system (8.1) in polar coordinates r, θ ($x = r\cos\theta, y = r\sin\theta$). We obtain the system

$$\begin{aligned} \dot{r} &= \varepsilon_1^2[(X_0 + X_1)\cos\theta + (Y_0 + Y_1)\sin\theta] \\ \dot{\theta} &= p + \varepsilon_1^2 r^{-1}[-(X_0 + X_1)\sin\theta + (Y_0 + Y_1)\cos\theta + \kappa r \varepsilon_1^{k-2}], \quad p \in \mathbf{N} \end{aligned} \tag{8.2}$$

in which

$$X_0(0, r\cos\theta, r\sin\theta)\cos\theta + Y_0(0, r\cos\theta, r\sin\theta)\sin\theta = Ar^3 \ (A = \text{const})$$

$$-X_0(0, r\cos\theta, r\sin\theta)\sin\theta + Y_0(0, r\cos\theta, r\sin\theta)\cos\theta = Br^3 \ (B = \text{const})$$

System (8.2) contains the parameter ε_1. When $\varepsilon_1 = 0$, system (8.2) permits of a $2\pi/p$-periodic motion ($p \in \mathbf{N}$)

$$r = r^0 \ (r^0 = \text{const}), \quad \theta = \theta_*(t) = pt + \theta^0, \quad \theta^0 = \text{const}$$

When $\varepsilon_1 \neq 0$, the problem of the existence of periodic motions is solved by the simple roots of the system of amplitude equations

$$\begin{aligned} &\int_0^{2\pi} [Ar^{0^3} + X_*(t)\cos\theta_*(t) + Y_*(t)\sin\theta_*(t)]dt = 0 \\ &\int_0^{2\pi} [Br^{0^3} - X_*(t)\sin\theta_*(t) + Y_*(t)\cos\theta_*(t) + \kappa^* r^0]dt = 0 \end{aligned} \tag{8.3}$$

$$X_*(t) = X_1(0, 0, 0, t) = a_0/2 + a_1\cos t + b_1\sin t + a_2\cos t + b_2\sin t + \dots$$
$$Y_*(t) = Y_1(0, 0, 0, t) = a_0^*/2 + a_1^*\cos t + b_1^*\sin t + a_2^*\cos t + b_2^*\sin t + \dots$$

($\kappa^* = \kappa$ when $k = 2$ and $\kappa^* = 0$ when $k > 2$).
We write Eqs (8.3) in the explicit form

$$Ar^{0^3} + (a_p + b_p^*)\cos\theta^0 + (a_p^* - b_p)\sin\theta^0 = 0$$

$$Br^{0^3} + \kappa^* r^0 + (a_p - b_p^*)\cos\theta^0 - (a_p^* + b_p)\sin\theta^0 = 0$$

In complex form, this system can be written in the form of the equation

$$(A+iB)r^{0^3}+\kappa^* r^0+Ce^{-i\theta^0} = 0, \quad C = (a_p+b_p^*)+i(a_p^*-b_p)$$

from which, when $\kappa^* = 0$, we find the unique root

$$r^{0^3} = |C/(A+iB)|, \quad \theta^0 = \arg(C/(A+iB)) \tag{8.4}$$

In the general case for a positive root $r^0 > 0$, we have

$$\begin{gathered} r^{0^3}+\frac{A\kappa^*}{A^2+B^2}r^0+\frac{|C|}{\sqrt{A^2+B^2}}\cos(\theta_1+\theta_2-\theta^0) = 0 \\ \kappa^* r^0+|C|\sqrt{A^2+B^2}\sin(\theta_1+\theta_2-\theta^0) = 0, \quad \theta_1 = \arg C, \quad \theta_2 = \arg(A-iB) \end{gathered} \tag{8.5}$$

We then find r^0 from the second equality and, on substituting it into the first equation, we find

$$r^{0^3} = -\sqrt{\frac{A^2+C^2}{A^2+B^2}}\cos(\theta_1+\theta_2+\theta_3-\theta^0), \quad \mathrm{tg}\theta_2 = \frac{A}{|C|}$$

Hence, a non-degenerate system ($C \neq 0$) of amplitude equations always has a simple root. This means that system (8.1) has 2π-periodic motion.

Theorem 9. For sufficiency small $|\mu| \neq 0$, the non-degenerate system (8.1) ($C \neq 0$) allows of a unique 2π-periodic solution of the form

$$x = \mu^{1/3}r^0\cos(pt+\theta^0)+o(\mu^{1/3}), \quad y = \mu^{1/3}r^0\sin(pt+\theta^0)+o(\mu^{1/3})$$

9. DEGENERATE OSCILLATIONS ACCOMPANYING A THIRD-ORDER RESONANCE

We assume that a system which has been linearized in the neighbourhood of zero has two frequencies. Then, when there is no "internal resonance" between the frequencies, oscillations of amplitude $O(\mu^{1/3})$ occur with respect to that pair of variables that corresponds to a frequency which is close to the frequency of the perturbating action, and the other pair has an amplitude $O(\mu)$. The situation changes in the case of an "internal" resonance when there is a relation of the form (2.1) in the system. Here, two pairs of variables now oscillate with an amplitude $O(\mu^\sigma)$ ($\sigma = 1/2$ for a third-order resonance and $\sigma = 1/3$ for second- and fourth-order resonances). In the treatment of these resonances, we will confine ourselves to a fourth-order system which corresponds to the indicated frequencies. In the case of an "internal" third-order resonance, the system in the complex-conjugate variables $\mathbf{z}$ and $\bar{\mathbf{z}}$ which has been normalized up to terms of the second order inclusive has the form

$$\begin{gathered} \dot{z}_s = i(3-2s)\omega_s z_s+(a_s+ib_s)\bar{z}_1^{s-1}\bar{z}_2^{3-s}+Z_{s0}(z,\bar{z})+\mu Z_{s1}(\mu,z,\bar{z},t) \\ \omega_1 = 2\omega_2+\kappa\varepsilon, \quad s = 1,2 \end{gathered} \tag{9.1}$$

(a_s and b_s are real constants and $Z_{s0} = O(||\mathbf{z}||^3)$).

We change the scale in system (9.1): $(\mathbf{z}, \bar{\mathbf{z}}) \to (\varepsilon_1\mathbf{z}, \varepsilon_1\bar{\mathbf{z}})$, $\varepsilon_1^2 = \mu$ and now make use of the polar coordinates r_s, θ_s. As a result, we obtain

$$\dot{r}_s = 2\varepsilon_1 R_s(\theta)\sqrt{r_1}r_2+\varepsilon_1 r_s^{1/2}(Z_s^* e^{-i\theta_s}+\bar{Z}_s^* e^{i\theta_s}), \quad s = 1,2 \tag{9.2}$$

$$\dot{\theta}_s = (3-2s)\omega_s+\varepsilon_1 Q_s(\theta)r_1^{s/2-1}r_2^{(3-s)/2}+\frac{\varepsilon_1}{2ir_s^{1/2}}(Z_s^* e^{-i\theta_s}-\bar{Z}_s^* e^{i\theta_s})$$

$$Z_s^* = \varepsilon_1 Z_{s0}^*+Z_{s1}^*, \quad Z_{s0}^* = \varepsilon_1^{-3}Z_{s0}(\varepsilon_1\sqrt{\mathbf{r}}e^{i\theta}, \varepsilon_1\sqrt{\mathbf{r}}e^{-i\theta})$$

$$Z_{s1}^* = Z_{s1}(\mu, \varepsilon_1\sqrt{\mathbf{r}}e^{i\theta}, \varepsilon_1\sqrt{\mathbf{r}}e^{-i\theta}, t), \quad s = 1,2; \quad \theta = \theta_1+2\theta_2$$

(the functions $R_s(\theta)$ and $Q_s(\theta)$ are defined in system (4.1)).

We will now calculate the functions

$$Z_{s1}^{**} = Z_{s1}(0,0,0,t) = X_{s1}^{*}(t) + iY_{s1}^{*}(t)$$
$$X_{s1}^{*} \equiv a_{0s}/2 + b_{1s}\sin t + a_{2s}\sin 2t + b_{2s}\cos 2t + \dots$$
$$Y_{s1}^{*} \equiv a_{0s}^{*}/2 + b_{1s}^{*}\sin t + a_{2s}^{*}\sin 2t + b_{2s}^{*}\cos 2t + \dots$$

and consider the resonance case when

$$\omega_1 = p + \kappa_1\varepsilon, \quad \omega_2 = p/2 - \kappa_2\varepsilon, \quad p \in \mathbf{N} \quad (\kappa_{1,2} = \text{const}, \ \varepsilon = \mu^k, \ k \ge 1/2)$$

It is clear that, when $\varepsilon_1 = 0$, system (9.2) has a $4\pi/p$-periodic solution

$$r_s = r_s^0, \quad s = 1, 2; \quad \theta_1^* = pt + \theta_1^0, \quad \theta_2^* = -pt/2 + \theta_2^0 \tag{9.3}$$

which depends on four arbitrary constants r_s^0, θ_s^0 and, when $p = 2q$, $q \in \mathbf{N}$, the solution will be $2\pi/p$-periodic.

In order to solve the problem of the existence in system (9.1) of a periodic solution for sufficiently small $|\mu| \ne 0$, we construct the system of amplitude equations [5]

$$\begin{gathered}\int_0^{2\pi}\left\{R_s(\theta^*)\sqrt{r_1^0 r_2^0} + \sqrt{r_s^0}[X_{s1}^*(t)\cos\theta_s^* + Y_{s1}^*(t)\sin\theta_s^*]\right\}dt = 0\\ \int_0^{2\pi}\left\{\kappa^*\kappa_s + Q_s(\theta^*){r_s^0}^{-1/2} r_2^0 + [-X_{s1}^*(t)\sin\theta_s^* + Y_{s1}^*(t)\cos\theta_s^*]\right\}dt = 0\end{gathered} \tag{9.4}$$

($\kappa^* = 1$ when $k = 1/2$, and $\kappa^* = 0$ when $k > 1/2$). We now write Eqs (9.4) in explicit form in each of the cases

1) $p = 2q, \quad q \in \mathbf{N}$

$$\begin{aligned}
&F \equiv 2(a_1\cos\theta^0 + b_1\sin\theta^0)r_2^0 + (a_{p1} + b_{p1}^*)\cos\theta_1^0 + (a_{p1}^* - b_{p1})\sin\theta_1^0 = 0\\
&\Phi \equiv 2\kappa^*\kappa_1\sqrt{r_1^0} + (-a_1\sin\theta^0 + b_1\cos\theta^0)r_2^0 -\\
&-[(a_{p1}^* - b_{p1})\cos\theta_1^0 - (a_{p1} + b_{p1}^*)\sin\theta_1^0] = 0\\
&2(a_2\cos\theta^0 + b_2\sin\theta^0)\sqrt{r_1^0 r_2^0} + (a_{q2} - b_{q2}^*)\cos\theta_2^0 + (a_{q2}^* + b_{q2})\sin\theta_2^0 = 0\\
&2\kappa^*\kappa_2\sqrt{r_2^0} + (-a_2\sin\theta^0 + b_2\cos\theta^0)\sqrt{r_1^0 r_2^0} +\\
&+(a_{q2}^* + b_{q2})\cos\theta^0 - (a_{q2} - b_{q2}^*)\sin\theta_2^0 = 0
\end{aligned} \tag{9.5}$$

2) $p = 2q - 1, \quad q \in \mathbf{N}$

$$\begin{aligned}
&F = 0, \quad \Phi = 0, \quad a_2\cos\theta^0 + b_2\sin\theta^0 = 0\\
&2\kappa^*\kappa_2 + (-a_2\sin\theta^0 + b_2\cos\theta^0)\sqrt{r_2^0} = 0
\end{aligned} \tag{9.6}$$

($\theta^0 = \theta_1^0 + 2\theta_2^0$).

Theorem 10. A periodic motion

$$z_s = \mu^{1/2}\sqrt{r_s^0}\exp\{i((3-2s)pt + \theta_s^0) + O(\mu^{1/2})\} + o(\mu^{1/2}), \ s = 1, 2 \tag{9.7}$$

of system (9.1) corresponds to each simple root of any of the systems of amplitude equations (9.5) and (9.6). When $p = 2q$, $q \in \mathbf{N}$, the period is equal to 2π and, when $p = 2q - 1$, $q \in \mathbf{N}$, the period is equal to 4π.

We will now analyses the conditions of Theorem 10. We first consider system (9.6). It can be seen that the subsystem consisting of the third and fourth equations does not have a solution when $r_2^0 \neq 0$ if $\kappa^* = 0$. This means that 2π-periodic motions can exist if the detuning of the resonance $\varepsilon = \mu^k, k > 1/2$. In this case $\operatorname{tg}\theta^0 = -a_2/b_2$, and r_2^0 is determined from the fourth equation of (9.6).

We $a_1/b_1 = -\operatorname{tg}\theta^0$, we obtain from the equation $F = 0$

$$\operatorname{tg}\theta_1^0 = -(a_{p1} + b_{p1}^*)/(a_{p1}^* - b_{p1})$$

and r_1^0 is uniquely determined from the condition $\Phi = 0$.

In the general case, when $a_1/b_1 \neq -\operatorname{tg}\theta^0$, we write the first two equations of system (9.6) in the form of a single complex equation

$$\begin{gathered} 2i\kappa^*\kappa_1\sqrt{r_1^0} + 2c_1 e^{-i\theta^0} r_2^0 + c_1^* e^{-i\theta_1^0} = 0 \\ c_1 = a_1 + ib_1, \quad c_1^* = (a_{p1} + b_{p1}^*) + i(a_{p1}^* - b_{p1}) \end{gathered} \tag{9.8}$$

We now determine $r_2^0(\theta^0)$ from the last equation of (9.6) and substitute it into Eq. (9.8). The radius r_1^0 is then uniquely calculated from (9.8).

In all the cases considered, system (9.6) admits of simple roots, and a 4π-periodic motion of the form of (9.7) exists.

In the case of a 2π-periodic motion, the first two equations of system (9.8) are written in the form (9.8). The remaining two equations are also conveniently represented in the complex form

$$\begin{gathered} 2i\kappa^*\kappa_2\sqrt{r_2^0} + 2c_2 e^{-i\theta^0}\sqrt{r_1^0 r_2^0} + c_2^* e^{-i\theta_2^0} = 0 \\ c_2 = a_2 + ib_2, \quad c_2^* = (a_{q2} - b_{q2}^*) + i(a_{q2}^* + b_{q2}) \end{gathered} \tag{9.9}$$

When $\kappa^* = 0$, system (9.8), (9.9) is easily analysed. This means that the problem of periodic motions in the general situation, when the detuning of the resonance is small ($\varepsilon = \mu^k, k > 1/2$), which also includes precise resonance, is solved.

When $\kappa^* = 1$, system (9.8), (9.9) is analysed by finding r_2^0 from (9.8) and the solution of Eq. (9.9), which now only contains r_1^0. Various special cases when one of the coefficients of (9.8) or (9.9) is equal to zero are examined.

10. FORCED OSCILLATIONS ACCOMPANYING A FOURTH-ORDER RESONANCE

We will assume that the system, when $\mu = 0$, is reduced to the normal form up to terms of the third order inclusive

$$\begin{gathered} \dot{z}_s = i(3-2s)\omega_s z_s + [C_{s1}|z_1|^2 + C_{s2}|z_2|^2]z_s + C_s^*\bar{z}_1^{s-1}\bar{z}_2^{4-s} + Z_{s0}(z,\bar{z}) + \mu Z_{s1}(\mu, z, \bar{z}, t) \\ C_{sj} = A_{sj} + iB_{sj}, \quad C_s^* = a_s + ib_s, \quad s, j = 1, 2; \quad \omega_1 = 3\omega_2 + \kappa\varepsilon \end{gathered} \tag{10.1}$$

($(A_{sj}, B_{sj}, a_s, b_s)$ are real coefficients and $Z_s = O(\|z\|^2)$). As in the treatment of a third-order resonance, we change the scale in system (10.1), by choosing $\varepsilon_1 = \mu^{1/3}$. Then, in polar coordinates r_s, θ_s, we have

$$\begin{gathered} \dot{r}_s = 2\varepsilon_1^2[(A_{s1}r_1 + A_{s2}r_2)r_s + R_s(\theta)r_1^{1/2}r_2^{3/2}] + \varepsilon_1^2 r_s^{1/2}(Z_s^* e^{-i\theta_s} + \bar{Z}_s^* e^{i\theta_s}) \\ \dot{\theta}_s = (3-2s)\omega_s + \varepsilon_1^2[B_{s1}r_1 + B_{s2}r_2 + Q_s(\theta)r_1^{s-3/2}r_2^{5/2-s}] + \frac{\varepsilon_1}{2ir_s^{1/2}}(Z_s^* e^{-i\theta_s} - \bar{Z}_s^* e^{i\theta_s}) \end{gathered} \tag{10.2}$$

$$R_s = a_s\cos\theta + b_s\sin\theta, \quad Q_s = -a_s\sin\theta + b_s\cos\theta, \quad \theta = \theta_1 + 3\theta_2; \quad s = 1, 2$$

(the functions Z_s^* have the same meaning as in system (9.2)). We will now consider the resonance case when

$$\omega_1 = p + \kappa_1\varepsilon, \quad \omega_2 = p/3 - \kappa_2\varepsilon, \quad p \in \mathbf{N} \quad (\kappa_{1,2} = \text{const}, \varepsilon = \mu^k, \kappa \geq 2/3)$$

Here, when $\varepsilon_1 = 0$, system (10.2) has a $6\pi/p$-periodic solution

$$r_s = r_s^0, \quad s = 1, 2; \quad \theta_1 = pt + \theta_1^0, \quad \theta_2 = -\frac{pt}{3} + \theta_2^0$$

(r_s^0, θ_s^0 are constants) and, when $p = 3q$, $q \in \mathbf{N}$, the solution will be be $2\pi/q$-periodic.
We will construct the systems of amplitude equations in each of the possible cases

$$\begin{aligned}
&1)\ p = 3q, \quad q \in \mathbf{N} \\
&F \equiv 2(A_{11}r_1^0 + A_{12}r_2^0)\sqrt{r_1^0} + R_1(\theta^0)r_2^{0^{3/2}} + (a_{p1} + b_{p1}^*)\cos\theta_1^0 + (a_{p1}^* - b_{p1})\sin\theta_1^0 = 0 \\
&\Phi = 2(B_{11}r_1^0 + B_{12}r_2^0)\sqrt{r_1^0} + Q_1(\theta^0)r_2^{0^{3/2}} + (a_{p1}^* - b_{p1})\cos\theta_1^0 - \\
&-(a_{p1} + b_{p1}^*)\sin\theta_1^0 + 2\kappa^*\kappa_1 r_1^{0^{1/2}} \\
&2(A_{21}r_1^0 + A_{22}r_2^0)\sqrt{r_2^0} + R_2(\theta)r_1^{0^{1/2}}r_2^0 + (a_{q2} - b_{q2}^*)\cos\theta_2^0 + (a_{q2}^* + b_{q2})\sin\theta_2^0 = 0 \\
&2(B_{21}r_1^0 + B_{22}r_2^0)\sqrt{r_2^0} + Q_2(\theta)r_1^{0^{1/2}}r_2^0 + (a_{q2}^* + b_{q2})\cos\theta_2^0 - \\
&-(a_{q2} - b_{q2}^*)\sin\theta_2^0 + 2\kappa^*\kappa_2 r_2^{0^{1/2}} = 0
\end{aligned} \tag{10.3}$$

$$\begin{aligned}
&2)\ p \neq 3q, \quad q \in \mathbf{N} \\
&F \equiv 0, \quad 2(A_{21}r_1^0 + A_{22}r_2^0) + R_2(\theta^0)\sqrt{r_1^0 r_2^0} = 0 \\
&\Phi \equiv 0, \quad 2(B_{21}r_1^0 + B_{22}r_2^0) + Q_2(\theta^0)\sqrt{r_1^0 r_2^0} + 2\kappa^*\kappa_2 r_2^{0^{1/2}} = 0
\end{aligned} \tag{10.4}$$

Theorem 11. A periodic solution

$$z_s = \mu^{1/3}\sqrt{r_s^0}\exp\{i((3-2s)pt + \theta_s^0) + O(\mu^{1/3})\} + o(\mu^{1/3}), \quad s = 1, 2 \tag{10.5}$$

of system (10.1) corresponds to each simple root of any of the systems of amplitude equations. In the case of (10.3), the period is equal to 2π, and, in the case of (10.4), the period is equal to 6π.

11. FORCED OSCILLATIONS ACCOMPANYING A SECOND-ORDER RESONANCE

We will write out the system in the complex-conjugate variables $\mathbf{z}$ and $\bar{\mathbf{z}}$

$$\begin{aligned}
\dot{z}_s &= i(3-2s)\omega_s z_s + [C_{s1}|z_1|^2 + C_{s2}|z_2|^2]z_s + [C_{s1}^*|z_1|^2 + C_{s2}^*|z_2|^2]\bar{z}_{3-s} + \\
&+ C_1^{**}\bar{z}_1^s\bar{z}_2^{3-s} + Z_{s0}(\mathbf{z}, \bar{\mathbf{z}}) + \mu Z_{s1}(\mu, \mathbf{z}, \bar{\mathbf{z}}, t), \quad s = 1, 2
\end{aligned} \tag{11.1}$$

Here, $Z_s = O\|\mathbf{z}\|^3)$, C_{sj}, C_{sj}^*, C_s^{**} are complex coefficients ($s, j = 1, 2$) and

$$\omega_1 = p + \kappa_1\varepsilon, \quad \omega_2 = p - \kappa_2\varepsilon, \quad p \in \mathbf{N} \quad (\kappa_{1,2} = \text{const}, \ \varepsilon = \mu^k, \ k \geq 2/3)$$

When $\mu = 0$, system (11.1) is already reduced to the normal form up to terms of the third order inclusive.
We change the scale: $(\mathbf{z}, \bar{\mathbf{z}}) \to (\varepsilon_1\mathbf{z}, \varepsilon_1\bar{\mathbf{z}})$, $\varepsilon_1 = \mu^{1/3}$ and then change to the polar coordinates r_s, θ_s ($s = 1, 2$). When $\varepsilon_1 = 0$, the resulting system admits of a $2\pi/p$-periodic solution

$$r_s = r_s^0, \quad s = 1, 2; \quad \theta_1 = pt + \theta_1^0, \quad \theta_2 = -pt + \theta_2^0$$

(r_s^0, θ_s^0 are constants). It is necessary to set up the system of amplitude equations

$$\begin{aligned}&(C_{s1}r_1^0 + C_{s2}r_2^0)\sqrt{r_s^0} + (C_{s1}^* r_1^0 + C_{s2}^* r_2^0)e^{-i\theta^0}\sqrt{r_{3-s}^0} + C_s^{**}\sqrt{r_s^0 r_{3-s}^0}e^{-2i\theta^0} + \\ &+ D_s e^{-i\theta_s^0} + 2i\kappa^*\kappa_s\sqrt{r_s^0} = 0 \\ &D_1 = (a_{p1} + b_{p1}^*) + i(a_{p1}^* - b_{p1}), \quad D_2 = (a_{p2} - b_{p2}^*) + i(a_{p2} - b_{p2}^*) \\ &\theta^0 = \theta_l^0 + \theta_2^0\end{aligned} \tag{11.2}$$

in order to solve the problem concerning the periodic solutions when $\varepsilon_1 \neq 0$.

Theorem 12. A 2π-periodic solution of the form (10.5) corresponds to each simple root (r_1^0, r_2^0, θ_1^0, θ_2^0), $r_s^0 > 0$ ($s = 1, 2$), of system (11.2).

12. APPLICATIONS

The theory of a cycle which has been described above finds application in the study of periodic motions and their families in the neighbourhood of equilibrium and the study of steady motions in various mechanical systems: conservative systems, Lyapunov systems and systems of general form (a heavy rigid body with a fixed point, a heavy rigid body on an absolutely rough plane, three-body problems, etc.).

We will outline the formulations of two applied problems.

Due to the constant small acceleration imparted to it, a geostationary satellite "hovers" all the time above the same point of the Earth's surface at a latitude φ [12]. The body of the satellite is at rest with respect to the orbital system of coordinates [13]. These steady conditions are described by the equations of translational-rotational motion, that is, by a twelfth-order system of general form. The results of an investigation of the linearized system in the neighbourhood of the steady motion for different latitudes have been presented earlier [13]. It is found that 19 third-order resonances occur when $\varphi = 2°$, of which seven are two-frequency resonances. The majority of the resonances are due to the interaction of the translational and rotational motions. Resonance curves have been presented and information has been given concerning the behaviour of the coefficients of the normal form [13].

A heavy, rigid body on an absolutely rough horizontal plane admits of a one-parameter family of permanent rotations of the following form: the body touches the supporting surface at one and the same point of the body and rotates about the vertical passing through this point with a constant angular velocity ω. At the same time, the centre of mass of the body describes a circle, which is parallel to the supporting plane, with its centre on the axis of rotation [14]. The body is described by an invertible system of differential equations [15]. The system allows of two first integrals, an energy first integral and geometric first integral [14].

The characteristic equation, corresponding to the rotations [14], has two zero roots. One of these is due to the one-parameter ω-family of permanent rotations and the second is due to the existence of a geometric integral. The remaining roots are determined from the equation

$$\kappa_0\mu^4 + \kappa_1\mu^3 + \kappa_2\mu^2 + \kappa_3\mu + \kappa_4 = 0$$

When $\omega = 0$ (equilibrium), we have $\kappa_1 = \kappa_3 = 0$. The reversibility of the system can be used in the investigation of the neighbourhood of equilibrium.

In the case of Celtic stone, we have $\kappa_1 \neq 0$, $\kappa_3 \neq 0$ ([14, page 122]). However, for a body in which $a_1 = a_2$ and/or $\sigma = 0 \pmod{\pi/2}$, we obtain $\kappa_1 = \kappa_3 = 0$ (a_1 and a_2 are the principal radii of curvature of the body surface at the point of contact of the body and the plane, and σ is the angle between the direction of the principal curvature corresponding to the radius a_1 and one of the principal axes of inertia).

When investigating the permanent rotations ($\omega \neq 0$), we use the integrals and we obtain a fourth-order system which depends on the constant of the integral. The reduced system can be reversible both in the case of a homogeneous ellipsoid [16] as well as in the case of a system of general form. In this case, when $\kappa_1 = \kappa_3 = 0$, the existence of a cycle is established by Theorems 2–5.

The above applied problems are important and each of them requires separate consideration.

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